MAT 562SE Differential Equations & Linear Algebra
Final Exam (9 Dec 2008)
Time allowed: 120 mins

Each question carries 10 marks

1. Let
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 & 3 & -1 \\ 0 & 0 & -2 & 0 & 7 \\ 2 & 4 & -10 & 6 & 12 \\ 2 & 4 & -5 & 6 & -5 \end{pmatrix}.$$

   Find

   (a) the rank of $\mathbf{A}$.

   (b) a basis for the row space of $\mathbf{A}$.

   (c) a basis for the column space of $\mathbf{A}$.

   (d) a basis for the null space of $\mathbf{A}$.

2. Let $L[y] = \dfrac{d^3y}{dt^3} - \dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} - 4y$.

   (a) Solve the homogeneous equation
$$L[y] = 0.$$

   (b) Set up the appropriate form of a particular solution Y_p of the non-homogeneous equation
$$L[y] = 3e^{2t} - e^t + \cos 2t,$$
   but do not determine the values of the coefficients.

3. Use the method of variation of parameters to solve
$$y'' - 4y = 4e^{2t}.$$

4. Let
$$\mathbf{A} = \begin{pmatrix} -1 & 6 & 3 \\ 3 & -4 & -3 \\ -6 & 12 & 8 \end{pmatrix}.$$
   (a) Diagonalize $\mathbf{A}$.
   (b) Find $\mathbf{A}^6$.
   (c) Express $\mathbf{A}^{-1}$ as a polynomial of $\mathbf{A}$.

5. Let
$$\mathbf{A} = \begin{pmatrix} 3 & -5 & -3 \\ 3 & -8 & -4 \\ -5 & 15 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$
   (a) Find $\exp At$.
   (b) Solve the initial value problem
$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

6. Let
$$\mathbf{A} = \begin{pmatrix} -5 & 9 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
   (a) Find a generalized eigenvector of rank 2 of $\mathbf{A}$.
   (b) Solve
$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t).$$
   (c) Set up the appropriate form of a particular solution $\mathbf{x}_p(t)$ of the non-homogeneous equation
$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), + \begin{pmatrix} t \\ e^{-2t} \end{pmatrix}$$
   but do not determine the values of the coefficient vectors.

7. Use the method of variation of parameters or otherwise to solve
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}.$$

— End of Paper —